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Conditional Lie–Bäcklund symmetry and reduction of evolution equations

R Z Zhdanov† ‡

Arnold-Sommerfeld Institute for Mathematical Physics, Leibnitzstraße 10, 38678 Clausthal-Zellerfeld, Germany

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Abstract. We suggest a generalization of the notion of invariance of a given partial differential equation with respect to a Lie–Bäcklund vector field. Such a generalization proves to be effective and enables us to construct principally new ansatz reducing evolution-type equations to several ordinary differential equations. In the framework of the said generalization, we obtain principally new reductions of a number of nonlinear heat conductivity equations $u_t = u_{xx} + F(u, u_x)$ with poor Lie symmetry and obtain their exact solutions. It is shown that these solutions cannot be constructed by means of the symmetry reduction procedure.

1. Introduction

Construction of exact solutions of nonlinear partial differential equations (PDEs) is one of the most important problems of modern mathematical physics. The most effective and universal method used is the symmetry reduction procedure pioneered by Sophus Lie. But there is a natural restriction on the application of the said procedure: the equations being studied should have non-trivial Lie symmetry. There exist very important equations (in particular, those describing heat conductivity and some nonlinear processes in biology) with very poor Lie symmetry. So, it would be desirable to modify the symmetry reduction procedure in such a way that it could be applied to these equations as well. Fortunately, the main idea of the symmetry reduction procedure—the reduction of the equation being studied to PDEs having less independent variables by means of specially chosen ansatz—can be applied to some of these if one utilizes their *conditional* symmetry (see [5, 7]). The method of conditional symmetries of PDEs is closely connected with the 'non-classical reduction' [1] and 'direct reduction' [2] methods (see also [12, 13]).

Further possibilities of constructing exact solutions of PDEs exist by the use of their Lie–Bäcklund (higher, generalized) symmetry [11]. In this way multi-soliton solutions of the KdV, mKdV, sine–Gordon and cubic Schrödinger equations can be obtained [3]. The choice of physically significant examples of equations admitting non-trivial Lie–Bäcklund symmetry is very restricted, however, there are examples due to Galaktionov *et al* [10, 16] and Fushchych *et al* [4, 6] of ansatz reducing PDEs admitting only trivial Lie–Bäcklund symmetry to systems of ordinary differential equations (ODEs). These facts can be understood within the framework of the *conditional* Lie–Bäcklund symmetry which is introduced below.

† On leave from the Institute of Mathematics of the Academy of Sciences of Ukraine, Tereshchenkivska Str. 3, 252004 Kiev, Ukraine.

‡ E-mail: asrz@pta3.pt.tu-clausthal.de

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It will be established that conditional invariance of the equation under study ensures its reducibility and this can be applied to construct its exact solutions. Since the class of PDEs conditionally invariant with respect to some Lie–Bäcklund field is substantially wider than the class of PDEs admitting Lie–Bäcklund symmetry in the classical sense, the said result yields principally new possibilities for the reduction of PDEs with poor Lie and Lie–Bäcklund symmetry. We will give several examples of reduction of PDEs to systems of ODEs by means of the ansatz corresponding to their conditional Lie–Bäcklund symmetry and we will show that the exact solutions obtained in this way cannot be constructed by means of the classical symmetry reduction procedure.

Let

$$u_t = F(t, x, u, u_1, u_2, \dots, u_n) \tag{1}$$

where $u \in C^n(\mathbb{R}^2, \mathbb{C}^1)$, $u_k = \partial^k u/\partial x^k$, $1 \leqslant k \leqslant n$, be some evolution-type equation and

$$Q = \eta\partial_u + (\mathrm{D}_x\eta)\partial_{u_1} + (\mathrm{D}_t\eta)\partial_{u_t} + (\mathrm{D}_x^2\eta)\partial_{u_2} + \cdots \tag{2}$$

with

$$\eta = \eta(t, x, u, u_t, u_1, u_{tt}, u_{t1}, \dots) \tag{3}$$

some smooth Lie–Bäcklund vector field (LBVF).

In the above formulae we denote the total differentiation operators with respect to the variables t and x by the symbols D_t and D_x respectively:

$$\mathrm{D}_t = \partial_t + u_t\partial_u + u_{tt}\partial u_t + u_{t1}\partial u_1 + \cdots$$
$$\mathrm{D}_x = \partial_x + u_1\partial_u + u_{t1}\partial u_t + u_2\partial u_1 + \cdots.$$

If the function η is of the form

$$\eta = \tilde{\eta}(t, x, u) - \xi_0(t, x, u)u_t - \xi_1(t, x, u)u_x \tag{4}$$

then the LBVF (2) is equivalent to the usual Lie vector field and can be represented in an equivalent form as

$$Q = \xi_0(t, x, u)\partial_t + \xi_1(t, x, u)\partial_x + \tilde{\eta}(t, x, u)\partial_u.$$

Definition 1. We say that equation (1) is invariant under the LBVF (2) if the condition

$$Q(u_t - F)|_M = 0 \tag{5}$$

holds.

In (5) M is a set of all differential consequences of the equation $u_t - F = 0$.

Definition 2. We say that equation (1) is conditionally invariant under LBVF (2) if the following condition

$$Q(u_t - F)|_{M \cap L_x} = 0 \tag{6}$$

holds.

Here, the symbol L_x denotes the set of all differential consequences of the equation $\eta = 0$ with respect to the variable x.

Evidently, condition (5) is nothing but the usual invariance criterion for equation (1) under LBVF (2) written in a canonical form (see, e.g. [11]). Most 'soliton equations' like the KdV, mKdV, cubic Schrödinger and sine–Gordon equations admit infinitely many LBVFs which can be obtained from some initial LBVF by applying the recursion operator.

Another important remark is that on the set of solutions of equation (1) we can exclude all derivatives with respect to t and thus obtain the vector field (2) with η of the form

$$\eta = \eta(t, x, u, u_1, u_2, \ldots, u_N). \tag{7}$$

In the following we will consider LBVFs of the form (2) and (7) only.

Clearly, if equation (1) is invariant under LBVF (2), then it is conditionally invariant under the said field; however, the inverse assertion is not true. This means, in particular, that definition 2 is a generalization of the standard definition of invariance of PDEs with respect to LBVF. Providing that (2) is a Lie vector field, definition 2 coincides with the definition of Q-conditional invariance under the Lie vector field.

One of the important consequences of Q-conditional invariance of a given PDE under the Lie vector field is the possibility of obtaining an ansatz which reduces this PDE to a single PDE with less independent variables (see, e.g. [7]). We will show that conditional invariance of the evolution-type equation (1) ensures its reducibility to N ODEs (N is the order of the highest derivative contained in η from (7)).

2. Reduction theorem

Consider the nonlinear PDE

$$\eta(t, x, u, u_1, \ldots, u_N) = 0 \tag{8}$$

as the Nth-order ODE with respect to variable x. Its general integral is written (at least locally) in the form

$$u = f(t, x, \varphi_1(t), \varphi_2(t), \ldots, \varphi_N(t)) \tag{9}$$

where $\varphi_j(t)$, $j = \overline{1, N}$ are arbitrary smooth functions. We will call expression (9) an ansatz invariant under LBVF (2) and (7).

Theorem 1. Let equation (1) be conditionally invariant under the LBVF (2) and (7). Then ansatz (9) invariant under LBVF (2) and (7) reduces PDE (1) to a system of N ODEs for functions $\varphi_j(t)$, $j = \overline{1, N}$.

Proof. We first prove that given the conditions of the theorem the system of PDEs

$$\begin{cases} u_t = F(t, x, u, u_1, \ldots, u_n) \\ \eta(t, x, u, u_1, \ldots, u_N) = 0 \end{cases} \tag{10}$$

is compatible.

Differentiating the first equation of (10) N times with respect to x, differentiating the second equation once with respect to t and comparing the derivatives u_{Nt} and u_{tN} we obtain the equality

$$D_x^N F = -(\eta_{u_N})^{-1}(\eta_t + \eta_u u_t + \eta_{u_1} u_{1t} + \cdots + \eta_{u_{N-1}} u_{N-1t})$$

or

$$D_x^N F = -(\eta_{u_N})^{-1}(\eta_t + \eta_u F + \eta_{u_1} D_x F + \cdots + \eta_{u_{N-1}} D_x^{N-1} F).$$

Consequently, providing that the condition

$$(\eta_t + \eta_u F + \eta_{u_1} D_x F + \cdots + \eta_{u_N} D_x^N F)|_{M \cap L} = 0 \tag{11}$$

holds identically, where L is the set of all differential consequences of the equation $\eta = 0$, then the system of PDEs (10) is in involution and its general solution contains N arbitrary complex constants $C_1,\ C_2,\ldots,C_N$ [15]. We will now prove that relation (11) follows from (6).

By considering (2), equality (6) can be rewritten in the form

$$\mathrm{D}_t\eta-\eta F_u-(\mathrm{D}_x\eta)F_{u_1}-\cdots-(\mathrm{D}_x^n\eta)F_{u_n}|_{M\cap L_x}=0$$

or

$$\mathrm{D}_t\eta|_{M\cap L_x}=0.$$

Since $\mathrm{D}_t\eta=\eta_t+\eta_u u_t+\eta_{u_1}u_{1t}+\cdots+\eta_{u_N}u_{Nt}$, the above equation reads

$$\eta_t+\eta_u u_t+\eta_{u_1}u_{1t}+\cdots+\eta_{u_N}u_{Nt}|_{M\cap L_x}=0$$

whence

$$\eta_t+\eta_u F+\eta_{u_1}\mathrm{D}_x F+\cdots+\eta_{u_N}\mathrm{D}_x^N F|_{M\cap L_x}=0. \tag{12}$$

Since the manifold $M\cap L$ is contained in the manifold $M\cap L_x$, relation (11) follows from relation (12).

Next, we consider the determinant

$$\Delta=\begin{vmatrix}\dfrac{\partial f}{\partial\varphi_1} & \dfrac{\partial f}{\partial\varphi_2} & \cdots & \dfrac{\partial f}{\partial\varphi_N}\\ \dfrac{\partial^2 f}{\partial\varphi_1\partial x} & \dfrac{\partial^2 f}{\partial\varphi_2\partial x} & \cdots & \dfrac{\partial^2 f}{\partial\varphi_N\partial x}\\ \vdots & \vdots & \ddots & \vdots\\ \dfrac{\partial^N f}{\partial\varphi_1\partial x^{N-1}} & \dfrac{\partial^N f}{\partial\varphi_2\partial x^{N-1}} & \cdots & \dfrac{\partial^N f}{\partial\varphi_N\partial x^{N-1}}\end{vmatrix}. \tag{13}$$

The determinant Δ is the Wronsky determinant for functions $y_j=\partial f/\partial\varphi_j,\ j=\overline{1,N}$. We will prove in the case considered that $\Delta\neq 0$.

Let $\Delta=0$, then due to the properties of the Wronsky determinant the functions y_j are linearly dependent. Consequently, there exist $\lambda_j=\lambda_j(t),\ j=\overline{1,N}$ such that

$$\sum_{j=1}^N\lambda_j(t)y_j=0.$$

Substituting $y_j=\partial f/\partial\varphi_j$ into the above equality we obtain

$$\sum_{j=1}^N\lambda_j(t)\frac{\partial f}{\partial\varphi_j}=0. \tag{14}$$

Integrating the first-order PDE (14) we have

$$f=\tilde f(t,x,\omega_1,\omega_2,\ldots,\omega_{N-1})$$

where $\omega_j=\lambda_N\varphi_j-\lambda_j\varphi_N,\ \ j=\overline{1,N-1}$. Consequently, in the case $\Delta=0$ the general solution of ODE (8) depends not on N but on $N-1$ arbitrary constants $\omega_j(t),\ j=\overline{1,N-1}$. We arrive at a contradiction, due to the assumption that $\Delta=0$. Hence, we conclude that $\Delta\neq 0$.

Substituting (9) into (1) we obtain

$$\sum_{j=1}^N\dot\varphi_j\frac{\partial f}{\partial\varphi_j}=-f_t+F\left(t,x,\frac{\partial f}{\partial x},\frac{\partial^2 f}{\partial x^2},\ldots,\frac{\partial^n f}{\partial x^n}\right)$$

or

$$\sum_{j=1}^{N} \dot{\varphi}_j \frac{\partial f}{\partial \varphi_j} = G(t, x, \varphi_1(t), \varphi_2(t), \ldots, \varphi_N(t)) \tag{15}$$

where an overdot means differentiation with respect to t.

Differentiation of (15) $N-1$ times with respect to the variable x yields

$$\sum_{j=1}^{N} \dot{\varphi}_j \frac{\partial^{k+1} f}{\partial \varphi_j \partial x^k} = \frac{\partial^k G}{\partial x^k} \qquad k = \overline{1, N-1}. \tag{16}$$

If we consider equations (15) and (16) as a system of linear inhomogeneous algebraic equations for functions $\dot{\varphi}_1,\ \dot{\varphi}_2, \ldots, \dot{\varphi}_N$, then its determinant has the form (13) and, consequently, is not equal to zero. Solving (15) and (16) with respect to the functions $\dot{\varphi}_j,\ j = \overline{1, N}$ we obtain

$$\dot{\varphi}_j = H_j(t, x, \varphi_1, \varphi_2, \ldots, \varphi_N) \qquad j = \overline{1, N}. \tag{17}$$

Let us expand the right-hand sides of (17) into a Taylor series with respect to the variable x in the neighbourhood of x_0 and then equate coefficients at $(x - x_0)^k$:

$$\dot{\varphi}_j = H_j(t, x_0, \varphi_1, \varphi_2, \ldots, \varphi_N) \qquad j = \overline{1, N} \tag{18}$$

$$0 = \frac{\partial^k H_j}{\partial x^k}(t, x_0, \varphi_1, \varphi_2, \ldots, \varphi_N) \qquad j = \overline{1, N},\ k \geqslant 1. \tag{19}$$

Thus, we have established that the system of PDEs (10) is equivalent to the infinite set of equations (18) and (19).

Next, we will prove that the right-hand sides of equations (19) vanish identically on the solutions of the system of ODEs (18).

Let $\varphi_j = \tilde{\varphi}_j(t,\ C_1,\ C_2, \ldots, C_N),\ j = \overline{1, N}$ where C_j are arbitrary complex constants, be a general solution of the system of ODEs (18). If at least one of the equations is not satisfied identically on the solutions of equations (18), then substituting into it the expressions for φ_j we obtain a relation of the form $h(C_1, C_2, \ldots, C_N) = 0$. Hence, it follows that the general solution of the system of PDEs (10) contains no more than $N-1$ independent constants. We arrive at a contradiction, which proves that the right-hand sides of equations (19) vanish identically on the solutions of system of ODEs (18). Consequently, system (18) and (19) is equivalent to the system of N ODEs

$$\dot{\varphi}_j = H_j(t, x_0, \varphi_1, \varphi_2, \ldots, \varphi_N) = \tilde{H}_j(t, x, \varphi_1, \varphi_2, \ldots, \varphi_N) \qquad j = \overline{1, N}. \tag{20}$$

Thus, *given the conditions of the theorem, ansatz* (9), *which is invariant under* LBVF (2) and (7), reduces equation (1) to the system of N ODEs (20) and the theorem is proved. □

Consequence. Let equation (1) be invariant under the LBVF (2) and (7). Then, ansatz (9) which is invariant under LBVF (2) and (7) reduces PDE (1) to a system of N ODEs for functions $\varphi_j(t),\ j = \overline{1, N}$.

The proof follows from the fact that if an equation is invariant under LBVF, then it is conditionally invariant with respect to this LBVF.

3. Some examples

Utilizing the above theorem, one can construct principally new exact solutions even for equations with poor Lie symmetry. As an illustration, we give several examples.

Example 1. Consider the nonlinear heat conductivity equation with a logarithmic-type nonlinearity

$$u_t = u_{xx} + (\alpha + \beta \ln u - \gamma^2 (\ln u)^2) u. \tag{21}$$

We will establish that equation (21) is conditionally invariant with respect to LBVF (2) with

$$\eta = u_{xx} - \gamma u_x - u^{-1} u_x^2. \tag{22}$$

Condition (6) for equation (21) reads

$$\mathrm{D}_t \eta - \mathrm{D}_x^2 \eta - (\alpha + \beta + (\beta - 2\gamma^2) \ln u - \gamma^2 \ln^2 u) \eta |_{M \cap L_x} = 0 \tag{23}$$

where M is the set of all differential consequences of equation (21) and L_x is the set of all differential consequences of the equation $u_{xx} - \gamma u_x - u^{-1} u_x^2 = 0$ with respect to x. Substituting expression (22) into the left-hand side of equation (23) and transferring to the manifold M (i.e. excluding the derivatives u_t, u_{tx}, u_{txx} with the help of equation (21)) we transform it to the form

$$2u^{-1}(u_{xx} - \gamma u_x - u^{-1} u_x^2)^2 + 4\gamma u^{-1} u_x (u_{xx} - \gamma u_x - u^{-1} u_x^2).$$

Evidently, the above expression does not vanish on the manifold M, but on the manifold $M \cap L_x$ it vanishes identically:

$$2u^{-1}(u_{xx} - \gamma u_x - u^{-1} u_x^2)^2 + 4\gamma u^{-1} u_x (u_{xx} - \gamma u_x - u^{-1} u_x^2)|_{M \cap L_x} = 0.$$

Hence, it follows that the nonlinear heat conductivity equation (21) is conditionally invariant under LBVF (2) with η of the form (22), but not invariant under the said LBVF in the sense of definition 1. This fact is also shown in [11], where the results of the classification of nonlinear heat conductivity equations $u_t = u_{xx} + F(u)$ admitting LBVF are given. It has been established, in particular, that only the linear heat equation admits an LBVF which cannot be represented in the form (2), (4) and, consequently, is not equivalent to a Lie vector field.

Integrating the equation $\eta \equiv u_{xx} - \gamma u_x - u^{-1} u_x^2 = 0$ as an ODE with respect to x we obtain an ansatz for $u(t, x)$:

$$u(t, x) = \exp(\varphi_1(t) + \varphi_2(t) \exp(\gamma x). \tag{24}$$

Substitution of ansatz (24) into equation (21) gives rise to a system of two ODEs:

$$\dot{\varphi}_1 = \alpha + \beta\varphi_1 - \gamma^2 \varphi_1^2 \qquad \dot{\varphi}_2 = (\beta + \gamma^2 - 2\gamma^2 \varphi_1)\varphi_2.$$

The general solution of the above system is given by one of the following formulae.

(i) $k = \beta^2 + 4\alpha\gamma^2 > 0$

$$u = C\left(\cos \frac{k^{1/2} t}{2}\right)^2 \exp(\gamma x + \gamma^2 t) + \frac{1}{2\gamma^2}\left(\beta - k^{1/2} \ \tan \frac{k^{1/2} t}{2}\right)$$

(ii) $k = \beta^2 + 4\alpha\gamma^2 < 0$

$$u = C\left(\cosh \frac{(-k)^{1/2} t}{2}\right)^2 \exp(\gamma x + \gamma^2 t) + \frac{1}{2\gamma^2}\left(\beta + (-k)^{1/2} \tanh \frac{(-k)^{1/2} t}{2}\right)$$

(iii) $k = \beta^2 + 4\alpha\gamma^2 = 0$

$$u = Ct^{-2}\exp(\gamma x + \gamma^2 t) + \frac{1}{2\gamma^2 t}(\beta t + 2).$$

Here, C is an arbitrary constant.

It is important to emphasize that the above solutions cannot be obtained by the symmetry reduction procedure. The maximal local invariance group of equation (21) is the two-parameter group of translations [14]

$$t' = t + \theta_1 \qquad x' = x + \theta_2 \qquad u' = u$$

and solutions (i) and (ii) are obviously not invariant under the above group.

Example 2. Consider the nonlinear heat conductivity equation

$$u_t = u_{xx} + F(u). \tag{25}$$

We will establish that it is conditionally invariant with respect to LBVF (2) with $\eta = u_{xx} - A(u)u_x^2$, providing that functions $F(u)$ and $A(u)$ satisfy the system of ODEs

$$\ddot{A} + 4A\dot{A} + 2A^3 = 0 \qquad \ddot{F} - \dot{A}F - A\dot{F} = 0. \tag{26}$$

Equality (6) for equation (25) takes the form

$$\mathrm{D}_t\eta - \mathrm{D}_x^2\eta - \dot{F}\eta|_{M\cap L_x} = 0$$

where M is the set of all differential consequences of the equation $u_t = u_{xx} + F(u)$, and L_x is the set of all differential consequences of the equation $u_{xx} - A(u)u_x^2 = 0$ with respect to x.

Excluding the derivatives u_t, u_{tx}, u_{txx} from the left-hand side of the above equality, and grouping terms in the obtained expression in a proper way, we have

$$2A\eta^2 + 4(\dot{A} + A^2)\eta + (\ddot{A} + 4A\dot{A} + 2A^3)u_x^4 + (\ddot{F} - \dot{A}F - A\dot{F})u_x^2|_{M\cap L_x} = 0$$

or taking equations (26) into considerations

$$2A\eta^2 + 4(\dot{A} + A^2)\eta|_{M\cap L_x} = 0. \tag{27}$$

Evidently, the left-hand side of equation (27) does not vanish on the manifold M but it does vanish on the manifold $M \cap L_x$. Consequently, the nonlinear heat equation (25) is conditionally invariant with respect to LBVF (2) with $\eta = u_{xx} - A(u)u_x^2$ if and only if equations (26) hold. Thus, the conditions of theorem 1 are satisfied and we can reduce equation (25) to two ODEs with the help of ansatz (9) invariant under the above mentioned LBVF.

Let the function $\theta(u)$ be determined by the equality

$$\int_0^{\theta(u)} (\ln\tau)^{-1/2}\,\mathrm{d}\tau = \alpha u + \beta$$

where α, β are arbitrary real constants. Then the ansatz

$$\int_0^{u(t,x)} \frac{\mathrm{d}\tau}{\theta(\tau)} = x\varphi_1(t) + \varphi_2(t)$$

reduce the nonlinear equation (25) with

$$F(u) = \left(\lambda_1 + \lambda_2\int_0^u \frac{\mathrm{d}\tau}{\theta(\tau)}\right)\theta(u) \tag{28}$$

to the system of ODEs

$$\dot{\varphi}_1 = \left(\frac{\alpha^2}{2}\varphi_1^2 + \lambda_2\right)\varphi_1 \qquad \dot{\varphi}_2 = \left(\frac{\alpha^2}{2}\varphi_1^2 + \lambda_2\right)\varphi_2 + \dot{\theta}(0)\varphi_1^2 + \lambda_1.$$

Here λ_1, λ_2 are arbitrary real constants and $\dot{\theta}(0)$ is a value of the first derivative of the function $\dot{\theta}(u)$ in the point $x = 0$.

The above system of ODEs is integrated in quadratures, thus giving rise to a family of exact solutions of the nonlinear PDE (25) with rather exotic nonlinearity (28). The solutions obtained are also non-invariant with respect to the two-parameter group of translations with respect to t and x, which is the maximal local invariance group of equation (25) and (28).

Example 3. Here, we will perform the reduction of a nonlinear PDE of the form (21):

$$u_t = u_{xx} + a(\ln^2 u)u \qquad a \in \mathbb{R}^1 \tag{29}$$

to systems of three ODEs.

By a rather cumbersome computation one can check that equation (29) is conditionally invariant with respect to LBVF (2) with

$$\eta = u^2 u_{xxx} - 3uu_x u_{xx} + 2u_x^3 + au_x u^2. \tag{30}$$

Integrating the third-order ODE $\eta = 0$ we obtain the following ansatz for the function $u(t, x)$.

(i) Under $a = \alpha^2 > 0$

$$u(t, x) = \exp\{\varphi_1(t) + \varphi_2(t)\cos\alpha x + \varphi_3(t)\sin\alpha x\} \tag{31}$$

(ii) under $a = -\alpha^2 < 0$

$$u(t, x) = \exp\{\varphi_1(t) + \varphi_2(t)\cosh\alpha x + \varphi_3(t)\sinh\alpha x\} \tag{32}$$

where φ_1, φ_2, φ_3 are arbitrary smooth functions.

Substitution of expressions (31) and (32) into PDE (29) gives rise to the following systems of nonlinear ODEs for the functions φ_1, φ_2, φ_3.

(i) Under $a = \alpha^2 > 0$

$$\begin{aligned}
\dot{\varphi}_1 &= \alpha^2(\varphi_1^2 + \varphi_2^2 + \varphi_3^2)\\
\dot{\varphi}_2 &= \alpha^2(2\varphi_1 - 1)\varphi_2\\
\dot{\varphi}_3 &= \alpha^2(2\varphi_1 - 1)\varphi_3
\end{aligned}$$

(ii) under $a = -\alpha^2 < 0$

$$\begin{aligned}
\dot{\varphi}_1 &= \alpha^2(\varphi_3^2 - \varphi_2^2 - \varphi_1^2)\\
\dot{\varphi}_2 &= \alpha^2(1 - 2\varphi_1)\varphi_2\\
\dot{\varphi}_3 &= \alpha^2(1 - 2\varphi_1)\varphi_3.
\end{aligned}$$

Making the change of the dependent variable $u = \exp v$, we rewrite equation (29) in the form

$$v_t = v_{xx} + v_x^2 + av^2 \tag{33}$$

and what is more, the ansatz (31) and (32) take the following form.

(i) Under $a = \alpha^2 > 0$

$$v(t, x) = \varphi_1(t) + \varphi_2(t)\cos\alpha x + \varphi_3(t)\sin\alpha x \tag{34}$$

(ii) under $a = -\alpha^2 < 0$

$$v(t, x) = \varphi_1(t) + \varphi_2(t)\cosh\alpha x + \varphi_3(t)\sinh\alpha x. \tag{35}$$

If we choose $\varphi_3 = 0$ in formulae (34) and (35), then the well known Galaktionov's ansatz are obtained [10, 16]. These ansatz were used to study blow-up solutions of the nonlinear PDE (33). It should be noted that all solutions of the nonlinear heat conductivity equations obtained in [10] can be constructed within the framework of our approach.

Example 4. Let us describe all PDEs of the form

$$u_t = u_{xx} + R(u, u_x) \tag{36}$$

which are conditionally invariant under LBVF (2) with $\eta = u_{xx} - au$, $a \in \mathbb{R}^1$.

Acting with the operator (2) on the equation (36) and transferring to the manifold $M \cap L_x$ we obtain the determining equation for the function R:

$$a^2u^2R_{u_xu_x} + 2auu_xR_{uu_x} + u_x^2R_{uu} + auR_u + au_xR_{u_x} + aR = 0.$$

The above PDE is rewritten in the form

$$(J^2 + a)R = 0$$

where $J = u_x\partial_u + au\partial_{u_x}$. This form is easily integrated and the general solution reads

$$R = f_1(u_x^2 - au^2)u_x + f_2(u_x^2 - au^2)u.$$

Here f_1, f_2 are arbitrary smooth functions.

Thus, the most general PDE of the form (36) conditionally invariant with respect to LBVF (2) with $\eta = u_{xx} - au$ is

$$u_t = u_{xx} + f_1(u_x^2 - au^2)u_x + f_2(u_x^2 - au^2)u. \tag{37}$$

Solving the equation $\eta \equiv u_{xx} - au = 0$ we obtain the following ansatz for $u(t, x)$.

(i) Under $a = -\alpha^2 < 0$

$$u(t, x) = \varphi_1(t)\cos\alpha x + \varphi_2(t)\sin\alpha x$$

(ii) under $a = \alpha^2 > 0$

$$u(t, x) = \varphi_1(t)\cosh\alpha x + \varphi_2(t)\sinh\alpha x.$$

These reduce PDE (37) to systems of two ODEs for functions $\varphi_1(t)$, $\varphi_2(t)$:

$$\dot\varphi_1 = -\alpha^2\varphi_1 + \alpha f_1^+\varphi_2 + f_2^+\varphi_1 \qquad \dot\varphi_2 = -\alpha^2\varphi_2 - \alpha f_1^+\varphi_1 + f_2^+\varphi_2$$
$$\dot\varphi_1 = \alpha^2\varphi_1 + \alpha f_1^-\varphi_2 + f_2^-\varphi_1 \qquad \dot\varphi_2 = \alpha^2\varphi_2 + \alpha f_1^-\varphi_1 + f_2^-\varphi_2$$

where $f_i^{\pm} = f_i(\alpha^2(\varphi_2^2 \pm \varphi_1^2))$.

4. Conclusion

In papers [8, 9] we constructed a number of ansatz of type (9) which reduce the nonlinear heat equation $u_t = [a(u)u_x]_x + f(u)$ to several ODEs. The basic technique used was the anti-reduction method. This paper provides a symmetry interpretation of these results. It is important to emphasize that there exist non-evolution equations which also admit anti-reduction. In particular, in [4, 6, 17] an anti-reduction of the nonlinear acoustics equation, of the equation for short waves in gas dynamics and of the nonlinear wave equation is carried out. It would be of interest to extend theorem 1 in order to consider these equations.

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